The Atom of the Mathematical "Reality" or the Composition and Decomposition of the Whole

***Simona Modreanu*[1]*, Alina Gavrilut*[2]*, Gabriel Crumpei*[3]*, Maricel Agop*[4]**

[1]Department of French language and literature, Alexandru Ioan Cuza University, Iasi, Romania, Email: simona.modreanu@gmail.com

[2]Department of Mathematics, Alexandru Ioan Cuza University, Iasi, Romania

[3]Department of Physics at Gh. Asachi Technical University, Iași, Romania

[4]Faculty of Psychology of Alexandru Ioan Cuza University in Iasi, Romania.

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In this paper, we aim at a transdisciplinary approach to atomicity. We especially focus on the mathematical perspective and we highlight the intimate, usual, defining property of the atom of being, in a sense, the essential indestructible, indivisible, irreducible, minimal, and self-similar unity. Using notions, concepts and results, we try to answer the question "What is the atom?" from a mathematical perspective, offering at the same time a series of possible interpretations and meanings that exceed its strict limits.

Keywords: Atomism; Transdisciplinarity; Atom; Pseudo-atom; Minimal atom; Fractality

In every atom of the realms of the universe
there exist vast oceans of system-worlds.
The Flower Adornment Sutra
(ancient Buddhist writing)

1 Introduction

The idea that matter is made up of discrete units is a very old one, which occurs in many ancient cultures, such as the Greek and Indian ones.

As a cosmological theory, atomism believes that the universe is composed of atoms that were thought to be indivisible and can never be transformed. This notion appeared in the years 500 BCE, its followers being the Greek philosophers Leucippus and Democritus, and the theory was developed later in the fourth century BCE by Epicurus. The Roman philosopher and poet Titus Lucretius Carus, in his *De rerum natura*, uses the synonym relation atomism - epicureanism. A theory from the times of Empedocles and Aristotle considered matter to be the element of origin of the universe, consisting of the four essential primordial elements: earth, fire, air and water. These ideas relied on philosophical and theological reasoning, rather than on concrete evidence and experiments.

Thus, Empedocles wrote:

> The elements predominate, taking turns, throughout a cycle, they disappear by merging with each other or continue to grow depending on the turn that fate assigns to them. Remaining the same, they circulate through each other [...]. So, in so far as one is born of the multiple and, conversely, after the decomposition of one, the multiple is reconstituted, all things appear and do not have a permanent duration. Because this perpetual change never stops, all things subsist through an unchanged cycle.

2 Axiomatization, Irreducibility and In(non)decomposability in Mathematics

In the modern understanding, a set of axioms is any collection of formally stated assertions from which other formally stated assertions follow by the application of certain well-defined rules.

Now, if we take the two sciences dealing with axiom, in logic, it is an indemonstrable first principle, rule, or maxim, that has found general acceptance or is thought worthy of common acceptance, while in mathematics, an axiom, postulate or assumption is a statement that is taken to be true, to serve as a premise or starting point from which other further reasoning, arguments and statements are logically derived. It is self-evidently true without proof.

Today, we take for granted that logic is the basis of intellectual activity. As a formal discipline, it started with Aristotle, who intended to catalog things (causes, events, animals, causes, etc.), identifying valid forms of arguments, and creating symbolic templates for them. This is what we basically considered as the the main content of logic for more than two thousand years.

But what is the support of logic itself?

If, in symbolic logic, we introduce symbols like p to stand for "proposition", then we can derive some basic "laws of thought" which are fundamental axiomatic rules upon which we base our rational discourse. The main logical rules are:

1. The law of identity: p is p;
2. The law of noncontradiction: p is not non-p;
3. The law of the excluded middle: either p or non-p.

But where do these laws of logic come from? Logic is a formal system. And it can be built on axioms, like Euclid's geometry. By the 1400's, algebra had been invented, and spread all over the world, bringing along cleaner symbolic representations of things. But it was only in 1847 that George Boole formulated logic in the same way as algebra, with logical operators like and/or, according to algebra-like rules. Then, in 1910, Whitehead and Russell (Principia Mathematica) generalized the idea that perhaps all of mathematics could be derived from logic.

Mathematicians have often used the words postulate and axiom as synonyms. Some recommend that the term axiom be reserved for the axioms of logic and postulate for those assumptions or first principles beyond the principles of logic by which a particular mathematical discipline is defined.

Mathematicians assume that axioms are true without being able to prove them. If someone starts with different axioms, then he will get a different kind of mathematics, but the logical arguments will be the same. Mathematics is not about choosing the right set of axioms, but about developing a framework from these starting points and every area of mathematics has its own set of basic axioms. In the early 20^{th} century, David Hilbert set up an extensive program to formalize mathematics and to resolve any inconsistencies in the foundations of mathematics. This included demonstrating all theorems using a set of simple and universal axioms, proving that this set of axioms is consistent, and proving that this set of axioms is complete, i.e. that any mathematical statement can be proved or disproved using the axioms.

Unfortunately, these plans were destroyed in 1931 by Kurt Gödel, who showed that in any (sufficiently complex) mathematical system with a certain set of axioms, one can find some statements which can neither be proved nor disproved using those axioms. When first published, Gëdel's theorems were deeply troubling to many mathematicians. When setting out to prove an observation, you do not know whether a proof exists – the result might be true but unprovable. Today we know that incompleteness is a fundamental part of logic, but also of computer science, which relies on machines performing logical operations. Surprisingly, it is possible to prove that certain statements are unprovable.

Axioms play a key role not only in mathematics but also in other sciences, notably in theoretical physics. In particular, the monumental work of Isaac Newton is essentially based on Euclid's axioms, augmented by a postulate on the non-relation of space-time and the physics taking place in it at any moment.

In 1905, Newton's axioms were replaced by those of Albert Einstein's special relativity, and later on by those of general relativity. Another paper of Albert Einstein and coworkers (see the Einstein-Podolski-Rosen paradox), almost immediately contradicted by Niels Bohr, concerned the interpretation of quantum mechanics. This was in 1935. According to Bohr, this new theory should be probabilistic, whereas according to Einstein it should be deterministic. The underlying quantum mechanical theory, i.e. the set of "theorems" derived by it, seemed to be identical. Einstein even assumed that it would be sufficient to add to quantum mechanics "hidden variables" to enforce determinism. However, thirty years later, in 1964, John Bell found a theorem, involving complicated optical correlations (see Bell inequalities), which yielded measurably different results using Einstein's axioms compared to using Bohr's axioms. And it took roughly another twenty years until an experiment of Alain Aspect got results in favor of Bohr's axioms, not Einstein's.

The role of axioms in mathematics and in the above-mentioned sciences is different. In mathematics one neither "proves" nor "disproves" an axiom for a set of theorems; the point is simply that in the conceptual realm identified by the axioms, the theorems logically follow. In contrast, in physics, a comparison with experiments always makes sense, since a falsified physical theory needs modification. Irreducibility refers to something that is incapable of being reduced or of being diminished or simplified further.

In mathematics, the concept of irreducibility is used in several ways. For instance, an irreducible polynomial is, roughly speaking, a polynomial that cannot be factored into the product of two non-constant polynomials. An irreducible fraction is a fraction in which the numerator and denominator are integers that have no other common divisors than 1 (and -1, when negative numbers are considered), while a fraction is reducible if it can be reduced by dividing both the numerator and denominator by a common factor. In topology, a hyperconnected space or irreducible space (the term irreducible space is preferred in algebraic geometry) is a topological space that cannot be written as the union of two proper closed sets (whether disjoint or non-disjoint). And the examples can go on; the essence remains the same...

Indecomposability (or, nondecomposability) refers to the incapacity of being partitioned, separated into components or basic elements. For instance, in abstract algebra, a module is indecomposable if it is non-zero and cannot be written as a direct sum of two non-zero submodules. In many situations, all modules of interest are completely decomposable; the indecomposable modules can then be thought of as the "basic building blocks", the only objects that need to be studied. In point-set topology, an indecomposable continuum is a continuum that is indecomposable, i.e. that cannot be expressed as the union of any two of its proper subcontinua. In probability theory, an indecomposable distribution is a probability distribution that cannot be represented as the distribution of the sum of two or more non-constant independent random variables. In chemistry and physics, atomic theory is a scientific theory of the nature of matter, which states that matter is composed of indivisible units called atoms.

3 Does matter exist?

It is easier to disintegrate an atom than a prejudice.
Albert Einstein

In the twentieth century, the term began to be used more and more by chemists in connection with the growing number of irreducible chemical elements. In the early twentieth century, through various experiments with electromagnetism and radioactivity, physicists discovered that the "indivisible" atom was actually a conglomeration of various subatomic particles (mainly electrons, protons and neutrons), which may exist separately; some others, such as Nobel laureate Max Planck, went as far as to state that matter does not exist. Thus, at the Nobel Prize in Physics in 1918, he said:

> As a man who has devoted his whole life to the most clearheaded science, to the study of matter, I can tell you as a result of my research about the atoms this much: There is no matter as such! All matter originates and exists only by virtue of a force which brings the particle of an atom to vibration and holds this most minute solar system of the atom together. We must assume behind this force the existence of a conscious and intelligent mind. This mind is the matrix of all matter.

Einstein supported this idea, namely that "the atom [is] energy rather than matter" and that "atomic energy is essentially mind-stuff".

Since atoms proved to be divisible, physicists later coined the term *elementary particles* in order to describe *the indivisible, though not indestructible, parts of an atom.*

It should be mentioned that the principle of indivisibility and that of irreducibility (indecomposability) are of overwhelming importance in mathematics. On the one hand, the atom, in the mathematical sense, does not model the physical atom, but instead preserves and reflects the idea of the atom in its indecomposable essence. To give just one example, a prime number is a natural number, greater than 1, which has exactly two divisors: the number 1 and the number itself. These divisors are improper. A prime number is therefore non-factorable, impossible to split, indivisible. The opposite of the notion of prime number is that of composite number.

On the other hand, mathematical modelling proves to be not only indispensable, but also in a fertile relationship of intertextuality with atomic physics. An article published in the journal *Notices of the American Mathematical Society* ("Math Unites the Celestial and the Atomic", Sept. 6, 2005) reports research that shows a hidden unity between the motion of objects in space and that of the smallest particles. The mathematics that describes celestial mechanics and that which governs certain aspects of atomic physics seem to correspond. The relationship between atomic dynamics and celestial dynamics seems to be underpinned by the same equations that characterize both the motion of bodies in celestial systems and the energy levels of electrons in simple systems.

Last but not least, in mathematical logic, an atomic formula or an atom designates a formula that does not contain its own sub-formulas. And this principle, of indivisibility, is also found in philosophy. For example, Leibniz's philosophical system is based on the existence of indivisible spiritual elements called *monads.* In Greek, *monas* means "unity", that which is "one". Leibniz believed that everything is, from a metaphysical point of view, reducible to a simple substance. Monads are simple substances that are not born and do not perish, their autonomy being total. They are indestructible, being a mirror of the entire universe. Monads hold a high degree of knowledge of the world, or rather a distinct, unique degree, which is due to the fact that they are eternal, without space, therefore they can be said to be indecomposable. Since "space is an illusion", in Leibniz's view, monads have no spatiality and therefore a distinction must be made between atoms and monads.

Therefore, according to Mandelbrot's principle of fractality (self-similarity), we can say that each monad is in itself a whole world, seen in a certain way. To put it differently, the world is made up of countless worlds.

4 Atomism and holism

We have seen, in brief, what atomism is. But as everything in this world exists in opposing pairs, the philosophical concept diametrically opposed to atomism, namely *holism*, was born. While atomism divides things in order to know them better, holism looks at things or systems as a whole and argues that in this manner we can know more about them, and understand better their nature and purpose.

Ian Smuts, who introduced the term *holism* in his 1926 book, *Holism and Evolution*, refers to the dialectic of the part-whole relationship as follows:

In the end it is practically impossible to say where the whole ends and the parts begin, so intimate is their interaction and so profound their mutual influence. In fact so intense is the union that the differentiation into parts and whole becomes in practice impossible, and the whole seems to be in each part, just as the parts are in the whole (1987/1926: 126).

The explanation is that, unlike atomism, which considers that any whole can be broken down by analyzing its separate parts and the relationships between them, holism states that the whole takes precedence and, due to the phenomenon of *emergence*, it is different and more important than the sum of its parts, as Nobel laureate Philip Anderson wrote in an article in the journal Science, to which he gave this very title, "More is different" (*Science* 04 Aug 1972: Vol. 177, Issue 4047, pp. 393-396, DOI: 10.1126/science.177.4047.393).

In fact one can argue that emergence explains the remarkable simplicity of complexity, and that the richness of the world around us emerges from the complex behaviour of many interacting components. As elegantly stated by the German scientist and engineer Jochen Fromm:

- one water molecule is not fluid
- one gold atom is not metallic
- one neuron is not conscious
- one amino acid is not alive.

Emergence therefore represents that property of a complex system, a property that arises from the collective interaction of its components and that cannot be predicted by analyzing only the structure or behaviour of a small number of constituents subject to fundamental laws.

When systems become very complex, it is almost impossible to predict their next states. Therefore, these phenomena require the introduction of new concepts and theories and, as this relates to the way we get to know the world, it has been called *epistemological emergence*, being associated in recent years with the complexity theory and with the nonlinear systems theory. Epistemological emergence does not threaten atomism at a fundamental level; it does not dispute the fact that the world is made, at ultimate level, of infinitesimal parts, but simply argues that the way these elements behave is not perfectly predictable.

In a logical approach, by holism we mean that the world functions in such a way that no part can be known without the whole being known first. Logical atomism is the alternative developed to logical holism, arguing that the world consists of logical facts (or atoms) that can no longer be broken down, and that each can be understood independently of other facts and without prior knowledge of the whole. All truths are thus dependent on a layer of atomic facts, and the world is made up of extremely simple and easy to understand facts.

The debates about the nature of reality, in which mathematics has always been involved, have brought it closer to the realm of quantum mechanics, even to that of psychology, where, with particular relevance and intensity, Nobel Prize-winning physicist Wolfgang Ernst Pauli and psychoanalyst Carl Gustav Jung, debated, over the course of an intense correspondence and a quarter of a century of friendship, about reality, the Self, the structure of the psyche, and the phenomena associated with it (e.g., synchronicity), aspects detailed in the book entitled "αATOM" (see www.theatlas.org).

The analogies between the structure of the psyche and that of the atom allowed Jung to formulate his theory of connection, the (inter) dependence of psychic and physical phenomena, thus structuring the ideas of the Self and the unconscious. While for the atom, the nucleus is the main source of energy, for the conscious Self the energy comes from the depths of the unconscious. This centre or nucleus represents for Jung the symbol of the totality of the psyche, which does not coincide with the Ego, but which has nevertheless always been perceived as external.

In this regard, Pauli admits that the field of physics becomes limited by materialism and considers that, beyond a certain point, it is about the existence of deeper spiritual layers, which cannot be adequately defined in conventional space-time terms.

Jung dwelled for a long time on the analogy that Pauli proposes between the atomic nucleus and the Self, and he wrote in the autumn of 1935:

> Generally speaking, the unconscious is thought of as psychic matter in an individual. However, the self-representation drawn up by the unconscious of its central structure does not accord with this view, for everything

> points to the fact that the central structure of the collective unconscious cannot be fixed locally but is an ubiquitous existence identical to itself; it must not be seen in spatial terms and consequently, when projected onto space, is to be found everywhere in that space. I even have the feeling that this peculiarity applies to time as well as space... A biological analogy would be the functional structure of a termite colony, possessing only unconscious performing organs, whereas the center, to which all the functions of the parts are related, is invisible and not empirically demonstrable. The radioactive nucleus is an excellent symbol for the source of energy of the collective unconscious, the ultimate external stratum of which appears an individual consciousness. As a symbol, it indicates that consciousness does not grow out of any activity that is inherent to it; rather, it is constantly being produced by an energy that comes from the depths of the unconscious and has thus been depicted in the form of rays since time immemorial [...]
>
> The center, or the nucleus, has always been for me a symbol of the totality of the psychic, as the conscious plus the unconscious, the center of which does not coincide with the ego as the center of consciousness, and consequently has always been perceived as being external.
> (C.G. Jung, *The Pauli/Jung Letters*, 1932-1958, Edited by C.A. Meier Princeton U. Press, Princeton, N.J., 2001).

Therefore, both come to the conclusion that living matter has a psychic aspect, and the psyche, a physical aspect. The quantum domain is the "place" where matter and the mind meet and where the laws of classical physics lead to nonlocal / acausal phenomena.

This is the realm of field theory, which describes how the single mind, *One*, links or connects all things (M.A. Fike, 2014). The suggestive phrase, "*Unus mundus*", used by Jung, reflects his beliefs about the unity of matter, mind, and spirit.

In our reality, we perceive this continuum as divided. The physicist David Bohm explains this in terms of *explicit order* or *unfolded order* (signifying the separation that takes place in the real world), respectively, *implicit order* or *enfolded order* (which describes the profound unit from which the physical world arises). The difference occurs between what is separate and local (explicit) and what is unified and nonlocal / acausal (implicit).

Quantum field theory therefore emphasizes the concept of holism, as one cannot consider a part without referring to its relationship to the whole. This theory reflects the concept of *entanglement* (a term introduced by Schrödinger), a phenomenon that explains "spooky action at a distance" (as Albert Einstein says) or nonlocality / acausality.

The division of reality into parts is the product of a convention, because subatomic particles, like all elements in the universe, are, in fact, connected to each other. In the general theory of relativity, Einstein states that space and time are not separate entities, but are instead connected to each other and represent part of a larger whole, which he called *the space-time continuum.* Bohm expanded on this idea, stating that everything in the universe is part of a continuum, and despite the apparent separation of things at the explicit level, the implicit and the explicit order merge into one another.

In their 1997 book on quantum physics, Marshall and Zohar wrote:

> In the quantum universe - and this means the entire universe - each part is "subtly" connected to any of the others, and the very identity - the being, qualities and characteristics - of constituents depends on the relations between them.

The meanings of part and whole, of atomism and holism can in fact coexist, depending on perspective. In his book *The Ghost in the Machine* (a title inspired by the Cartesian mind-brain dualism), Arthur Koestler proposed the term *holon* in order to designate something that is to an equal extent part and whole.

For instance, referring to life sciences, we can describe an atom as a holon - a whole entity made up of smaller parts such as protons, neutrons and electrons. The atom itself can be part of a larger holon, such as a molecule (several atoms chained together to form a new entity). In its turn, the molecule is a holon that can be part of a cell, or part of an organ, or part of a human being. Thus, holons have a dual nature: they are equally complete, autonomous entities, but they are also parts of other wholes. From this perspective, holons exist simultaneously as independent integers in relation to their component parts, but also as subordinate, dependent parts, subsystems of other holon systems.

Regarding the relations between part and whole, between *One* (the principle, the critical substance underlying the world) and *Multiple* (the diversity of phenomena), they have been described by mereology, the branch of logic developed by the Polish logician Stanislaw Lésniewski.

Since the beginnings of philosophy and, more recently, in mereology (Hudson, 2004), profound and difficult questions have arisen regarding the problem of atomism (mereologically, an atom is an entity that does not contain its own parts): Are there such entities? And if they exist, is everything entirely made up of atoms?

The two main options, in the sense that either there are no atoms at all, or that everything is ultimately made up of atoms, are reflected in two postulates that are mutually incompatible (Varzi, 2007).

Starting from an axiomatization, one might hope that a certain axiomatized mereological theory can ensure (decide) the existence of an atomic "space" (or domain), that is, it can guarantee that everything in this domain is composed only of atoms. The results obtained so far at this abstract level of axiomatization deny the existence of such an atomic domain (Tsai, 2017).

5 The mathematical "atom"

In the following, we shall see that the mathematical perspective preserves the intimate, defining property of the atom, in its various forms and mathematical meanings of being, in a sense, the essential indestructible, indivisible, irreducible, minimal and self-similar unity.

Using notions, concepts and results, we shall try to answer the question "What is the atom?" from a mathematical perspective, offering at the same time a series of possible interpretations and meanings that exceed its strict limits.

We emphasize that an atom is a mathematical object (an entity) that, in essence, has no other subobjects (subentities) than the object itself or the null subobject. The idea is also found in computer science, for example. Thus, in database systems, an *atomic transaction* is an indivisible and irreducible series of database operations, so that either all of them occur or nothing happens.

5.1 Elements of set theory

By a *set* X we mean a collection (an ensemble) of distinct objects (the elements of the set), which is well-determined and considered as an entity.

$\emptyset$ denotes the void set (or the empty set), so it does not contain any elements. In the following, we shall assume that the abstract, arbitrary set X (the space where we operate) is nonvoid, meaning it contains at least one element.

For instance, $\mathbb{N}$ denotes the set of all naturals $(0, 1, 2, 3, ...)$, $\mathbb{Z}$, the set of all integers $(..., -3, -2, -1, 0, 1, 2, 3...)$, $\mathbb{Q}$, the set of rationals (a rational number is a number that can be expressed as the quotient or fraction p/q of two integers, a numerator p and a non-zero denominator q), $\mathbb{R}\backslash\mathbb{Q}$, the set of all irrationals (which are not rational numbers, that is, they cannot be expressed as the ratio of two integers: $\sqrt{2}, \sqrt{3}, e, \pi$ etc.), $\mathbb{R}$, the set of all real numbers, that is, rational and irrational numbers.

All these numbers represent features of the surrounding world, *they reflect reality, but they are not part of it.*

By $\mathbb{R}_+$ we denote the set of all positive real numbers, that is, the subset of those real numbers that are greater than zero: $\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\}$. By $x > 0$ we mean $x \geq 0$ and $x \neq 0$.

Let A, B be two arbitrary sets from the abstract space X.

We say that *the set A is included in the set B* and we denote it by $A \subseteq B$ (or $B \supseteq A$), if any element x of A (denoted by $x \in A$) also belongs to the set B.

Obviously, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

If an element x does not belong to a set A, then we denote this by $x \notin A$.

By $A \subsetneq B$ (or $B \supsetneq A$) we mean that $A \subseteq B$ and $A \neq B$.

The symbol "$\forall$" means "for every", "$\exists$", "there exists", "$\nexists$, "it does not exist", "$\exists!$", "there exists and it is unique", and "$\Leftrightarrow$", "if and only if".

By $\mathcal{P}(X)$ we denote the family of all subsets of the set X, and by $\mathcal{P}_0(X)$, we mean the family of all nonvoid subsets of X.

For instance, if $X = \{1, 2, 3\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The union of the sets A and B is the set $A \cup B = \{x \in X; x \in A \text{ or } x \in B\}$.

The intersection of the sets A and B is the set $AnB = \{x \in X; x \in A \text{ and } x \in B\}$.

We say that two sets A, B are *disjoint* if $A \cap B = \emptyset$.

We say that a family $\{A_i\}_{i\in\{1,2,...,p\}}$ of nonvoid sets is a *partition of a set* A if $\cup_{i=1}^{p} A_i = A$ and the sets A_i are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$, $\forall i, j \in \{1, 2, ..., p\}$, $i \neq j$.

The difference of the sets A and B is the set $A \backslash B = \{x \in X; x \in A \text{ and } x \notin B\}$.

The symmetric difference of the sets A and B is the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

The complement of the set A is the set $cA = \{x \in X;\ x \notin A\}$.

The Cartesian product of the sets A and B is the set $A \times B = \{(a, b);\ a \in A, b \in B\}$, where (a, b) signifies the pair of elements $a \in A$ and $b \in B$ (in this order) (called ordered pair).

If $a, b \in \mathbb{R}$, then we introduce the following sets:
$(-\infty, \infty) = \mathbb{R}$,
the open interval $(a, b) = \{x \in \mathbb{R},\ a < x < b\}$,
the closed interval $[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$,
the interval open in a and closed in $b : (a, b] = \{x \in \mathbb{R},\ a < x \leq b\}$,
the interval closed in a and open in $b : [a, b) = \{x \in \mathbb{R},\ a \leq x < b\}$,
the open interval $(a, \infty) = \{x \in \mathbb{R},\ a < x\}$,
the closed interval $[a, \infty) = \{x \in \mathbb{R},\ a \leq x\}$,
the open interval $(-\infty, a) = \{x \in \mathbb{R},\ x < a\}$,
the closed interval $(-\infty, a] = \{x \in \mathbb{R},\ x \leq a\}$.

By *the modulus* of a real number x, denoted by $|x|$, we mean
$\max\{x, -x\} \left(= \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases} \right)$.

We say that a *set* A is *finite* if it contains a finite number of elements. We denote by $cardA$, the number of elements which constitute the set.

By a *function* $f{:}A \to B$ (defined on A and taking values in B), we mean an application (or, a relation, a law or a correspondence) so that *to any element* of A corresponds a *unique* element of $B : \forall x \in A$, $\exists! y = f(x) \in B$. Sometimes, the function f can be also expressed as $x \to f(x)$.

x is called the *argument of the function*, A is called the *domain of* f and B, the *codomain* of f (*the set of all values of* f).

We say that the *function* f is *increasing* (*decreasing*, respectively) if its y-values increase (decrease, respectively) as the x-value increase: $\forall x_1, x_2 \in A$, with $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$ $(f(x_1) \leq f(x_2))$, respectively.

A *sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements* of the space X is a function $f : \mathbb{N} \to X$ which associates to any natural $n \in \mathbb{N}$ a unique element $x_n \in X$, called the *general term of the sequence.*

If the following condition is satisfied: $\forall y \in B$, $\exists! x \in A$ so that $y = f(x)$ (any element of B is the image of a unique element from A), then the *function* f is called a *bijection* (or a *biunivocal application*).

We say that:

(i) two *sets* A and B are *equipotent* if there is a bijection between them;

(ii) a set A is said to be *countable* if A and the set $\mathbb{N}$ of all naturals are equipotent (in other words, the elements of the set A can be counted, and this process goes on to infinity);

(iii) a set A is said to be *at most countable* if it is either finite or countable.

We call the *cardinal of the set* A (denoted by $cardA$), the class (or the family) of all sets that are equipotent with a certain given set A. If the set is finite, then its cardinal represent the number of the elements of the set. If two sets are equipotent, then it is said that they have *the same cardinal.*

For instance, the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, the set of all even numbers and the set of all odd numbers are countable sets.

There are also sets that are not at most countable (called *uncountable*), for instance, $\mathbb{R} \setminus \mathbb{Q}, \mathbb{R}$ etc.

The countable and the uncountable sets are called *infinite sets.*

Cantor denoted card$\mathbb{N}$ by $\aleph_0$ (alef-zero, alef being the first letter in the Hebrew alphabet).

card$\mathbb{R}$, the cardinal of all real numbers, is called *the continuum* and it is denoted by c. Also, $\mathbb{R} \setminus \mathbb{Q}$ and any interval of the form $[a, b]$ have the cardinal c.

Returning to operations with sets, we can consider the union (intersection, respectively), of three sets: $A \cup B \cup C$ ($A \cap B \cap C$, respectively) and then the finite union (intersection, respectively): $A_0 \cup A_1 \cup ... \cup A_n \overset{denoted}{=} \cup_{i=0}^{n} A_i$ ($A_0 \cap A_2 \cap ... \cap A_n \overset{denoted}{=} \cap_{i=0}^{n} A_i$, respectively).

Recurrently, we can introduce countable unions (intersections, respectively) of sets: $A_0 \cup A_2 \cup ... \cup A_n \cup ... \overset{denoted}{=} \cup_{n=0}^{\infty} A_n$ ($A_0 \cap A_2 \cap ... \cap A_n \cap ... \overset{denoted}{=} \cap_{n=0}^{\infty} A_n$, respectively).

$(A_n)_{n \in \mathbb{N}}$ denotes the corresponding sequence of sets (Precupanu, 1998).

5.2 Order relations

Let X and Y be two arbitrary, nonvoid sets. A subset R of the Cartesian product $X \times Y$ is called a *relation from X to Y*. If, particularly, $X = Y$, we say that R is a *relation on X*.

If X is a nonvoid, abstract set, then a relation on X, denoted by "$\leq$" is called a *partial order relation* if the following axioms are fulfilled:

(i) *reflexivity*: $\forall x \in X$, $x \leq x$ (in other words, any element is in relation with itself or, each element is comparable with itself);

(ii) *antisymmetry*: $\forall x, y \in X$, from $x \leq y$ and $y \leq x$ it results $x = y$ (or, no two different elements precede each other);

(iii) *transitivity*: $\forall x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (or, the start of a chain of precedence relations must precede the end of the chain).

In this case, the set X endowed with the partial order relation R (denoted by (X, R)) is called a *(partially) ordered set.*

For instance, the relation "$\leq$" on $\mathbb{R}$ is an order relation. Also, the relation of inclusion of sets is a (partial) order relation.

A partially ordered set A is said to be *totally ordered* if $\forall x, y \in X$, it holds either $x = y$ or $y = x$ (in other words, any two of its elements are comparable).

Let $(X, \leq)$ be an arbitrary partially ordered set and A a nonvoid subset of X.

(i) An element $\alpha \in A$ is called *the greatest element* of A if $x = \alpha$, $\forall x \in A$ (an element of A that is greater than every other element of A);

(ii) An element $\beta \in X$ is called *the least element* of A if $\beta \leq x$, $\forall x \in A$ (an element of S that is smaller than every other element of S);

The greatest element of a partially ordered subset must not be confused with the maximal elements of the set. A set can have several maximal elements without having a greatest element.

(iii) An element $a \in A$ is called a *maximal element* of the set A if from the conditions $x \in A$ and $a = x$ it follows $x = a$ (it is an element of A that is not smaller than any other element in A);

(iv) An element $b \in A$ is called *a minimal element* of the set A if from the conditions $x \in A$ and $x \leq b$ it folllows that $x = b$ (it is an element of A that is not greater than any other element in A);

(v) An *upper bound* (or *majorant*) of A is an element $\beta \in X$ so that $a \leq \beta$, $\forall a \in A$ (is greater than or equal to every element of A).

(Dually, a *lower bound* or *minorant* of A is defined to be an element of X which is less than or equal to every element of A.)

Zorn's lemma. Every partially ordered set $(A, \leq)$ for which every totally ordered subset has an upper bound contains at least one maximal element.

The notion of an atom is found in algebra in the following sense:

An element a of a partially ordered set A possessing the property that there exists $0 \in A$ so that $0 = x$ for every $x \in A$ is called an *atom* if $0 < a$ and it does not exist $x \in A$ so that $0 < x < a$.

In fact, in partially ordered sets, atoms are generalizations of the singletons (that is, sets containing only one element) of the sets theory. Moreover, in this sense, atomicity (the property of a mathematical object of being atomic), provides a generalization in an algebraic context of the possibility of selecting an element from a nonempty set (Davey and Priestley, 2002).

In fact, in mathematical logic, *an atomic formula* is a formula without a deep propositional structure, that is, a formula that does not contain logical connections, or, equivalently, a formula that does not have strict subformulas.

Atoms are thus the simplest well-formed formulas of logic, the compound formulas being formed by combining atomic formulas using logical connections. Also, also in logic, an atomic sentence is a type of declarative sentence that is either true or false and that cannot be broken down into other simpler sentences.

In some models of set theory, an atom is an entity (a mathematical object) that can be an element of a set but does not itself contain elements with similar properties (hence the "ultimate" character of an atom).

As we shall see in the following, in mathematical analysis, a set's property of being an atom is defined in relation to another mathematical object, namely, with respect to a set (multi)function.

5.3 Elements of measure theory

Classes of sets

Let us consider an abstract, nonvoid set T and let be the family $\mathcal{P}(T)$ of all parts (subsets) of T.

O nonvoid subfamily $\mathcal{C} \subset \mathcal{P}(T)$ of subsets of T is called a *ring* if the following conditions (axioms) are fulfilled:

(i) $\forall A, B \in \mathcal{C}$ it follows that $A \setminus B \in \mathcal{C}$;

(ii) $\forall A, B \in \mathcal{C}$ it follows that $A \cup B \in \mathcal{C}$.

For instance, $\mathcal{P}(T)$ is obviously a ring of sets (the largest one in the sense of inclusion) of T.

If, additionally to the axioms (i) şi (ii), the following condition is fulfilled:

(iii) $T \in \mathcal{C}$,

then $\mathcal{C}$ is called an *algebra* of subsets of T.

For instance, $\mathcal{P}(T)$ is an algebra of sets (the largest in the sense of inclusion) of T.

A class $\mathcal{A}$ of subsets of T is said to form a $\sigma-$algebra if $\mathcal{A}$ is an algebra which satisfies, additionally, the property:

(iv) $\forall (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ (for every sequence of sets of $\mathcal{A}$) it holds that $\cup_{n=0}^{\infty} A_n \in \mathcal{A}$ (Precupanu, 2006).

Set functions

Let $\mathcal{C}$ be a ring of subsets of a non-empty abstract set T and $m : \mathcal{C} \to \mathbb{R}_+$ be a set function which satisfies the condition $m(\emptyset) = 0$.

The following notions generalize the notion of a measure in its classic sense (as a foundation of the field of mathematics, known as "measure" theory, a subdomain of mathematical analysis).

In mathematical analysis, a measure (in classic sense) is a function which "measures", assigning to certain sets of a class (family) of sets, a positive real number or $+\infty$. In this sense, a measure is a generalization of the concepts of length, area or volume.

One particularly important example is the Lebesgue measure on a Euclidean space, which assigns the conventional length, area and volume of Euclidean geometry to appropriate subsets of the Euclidean space $\mathbb{R}^n$. For instance, the Lebesgue measure of the interval $[0, 1]$ is its length in the ordinary sense of the word, namely, 1 (Precupanu, 2006; Royden, 1988; Fremlin, 2000).

A measure must be additive, which means that the measure of a set representing the union of a finite (or countable) number of smaller sets that are pairwise disjoint is equal to the sum of the measures of these smaller subsets.

The notions that we shall introduce next have contributed (among many others) to the development in recent years of the theory of non-additive measures, sometimes known as the fuzzy measures theory (Pap, 1995). These notions prove their utility due to the necessity to model phenomena from the real world, in circumstances in which the condition of additivity (either finite or countable), as an immediate property of a measure, is much too restrictive.

The set function m is called:
(i) *null-additive* if $m\,(A \cup B) = m\,(A)$, for every sets $A, B \in \mathcal{C}$, satisfying the condition $m(B) = 0$;
(ii) *null-null-additive* if $m(A \cup B) = 0$, for every sets $A, B \in \mathcal{C}$, satisfying the condition $m(A) = m(B) = 0$;
(iii) *diffused* if $m(\{t\}) = 0$, whenever $\{t\} \in \mathcal{C}$;
(iv) *monotone* if $m(A) = m(B)$, for every sets $A, B \in \mathcal{C}$, so that $A \subseteq B$;
(v) *null-monotone* if for every two sets $A, B \in \mathcal{C}$, having the property that $A \subseteq B$, if $m\,(B) = 0$ holds, then one necessarily has also $m(A) = 0$;
(vi) *finitely additive* if $m(A \cup B) = m(A) + \nu(B)$, for every disjoint sets $A, B \in \mathcal{C}$;
(vii) *subbaditive* if $m(A \cup B) = m(A) + \nu(B)$, for every (disjoint or not) $A, B \in \mathcal{C}$.

Example. (i) Let us suppose that $T = \{t_1, t_2, ..., t_n\}$, where for every $i \in \{1, 2, ..., n\}$, t_i represents a particle, and $m : \mathcal{P}\,(T) \to \mathbb{R}_+$ is a set function representing the mass of the particle.

In the macrosopic world, m is a finitely additive set function. At quantum scale, however, this statement no longer remains valid due to the phenomena of annihilation. For instance, if t_1 and t_2 represents an electron and a positron, respectively, then $m(\{t_1\}) = m(\{t_2\}) = 9,11 \times 10^{-31}$kg, but $m(\{t_1, t_2\}) = m(\{t_1\} \cup \{t_2\}) = 0$;

(ii) Entropy in Shannon's sense is a subadditive set function, taking real values (Gavriluţ and Agop, 2016; Gavriluţ, 2019).

sectionTypes of atoms

All things are made up of atoms
Richard Feynman

In the following, we shall present several types of atoms in their mathematical meaning, we shall establish some relationships among these types of atoms and we shall also highlight several possible interpretations.

Unless stated otherwise, $\mathcal{C}$ will represent a ring of subsets of an arbitrary nonvoid set T and $m : \mathcal{C} \to \mathbb{R}_+$, an arbitrary set function satisfying the condition $m(\emptyset) = 0$. As we have already seen, this abstract set function represents the generalization of the classic notion of a *measure* used in the domain of mathematics called "measure theory" and it is the mathematical object through which the process of so-called "measurement" is performed.

5.4 Atoms and pseudo-atoms

These are the main types of atoms from a mathematical perspective:

I. A set $A \in \mathcal{C}$ is called an *atom* of m if $m\,(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, it holds either $m(B) = 0$ or $m(A \setminus B) = 0$.

We observe that, in a certain sense, an atom is a special set, of strictly positive "measure", having additionally the property that any of its subsets either has zero "measure", or the difference set between the initial set and its subset we refer to has zero "measure".

An atom can be interpreted, from a physics viewpoint, as the correspondent of a black hole.

II. The set function m is said to be *non-atomic* if it has no atoms, that is, for every set $A \in \mathcal{C}$ with $m(A) > 0$, there exists a subset $B \in \mathcal{C}$ $(B \subseteq A)$ so that $m(B) > 0$ and $m(A \setminus B) > 0$.

III. A set $A \in \mathcal{C}$ is called a *pseudo-atom* of m if $m\,(A) > 0$ and for every subset $B \in \mathcal{C}$ $(B \subseteq A)$ one has either $m(B) = 0$ or $m(B) = m(A)$.

In other words, a pseudo-atom is a special set, of strictly positive "measure", for which any of its subsets either has null "measure", or has the same "measure" as the set itself. Thus, it can be stated that a pseudo-atom possesses the property that any of its subsets either has null "measure" (that is, it is *negligible during the "measurement" process), or it entirely "covers" the set (during the same "measurement" process).*

In other words, assuming that the set function m is monotone, then a pseudo-atom is a set of strictly positive "measure" and which does not contain any proper subset of strictly smaller and strictly positive "measure".

IV. The set function m is said to be *non-pseudo-atomic* if it does not have pseudo-atoms, that is, for any set $A \in \mathcal{C}$ with $m(A) > 0$, there exists a subset $B \in \mathcal{C}$ $(B \subseteq A)$ so that $m(B) > 0$ and $m(B) \neq m(A)$.

For instance, the Lebesgue measure on the real line is a measure (in the classic sense) which is non-pseudo-atomic (Royden, 1988), and therefore it does not have any pseudo-atom.

The non-pseudo-atomic measures satisfy the following remarkable property, which we owe to Sierpinski, a property which states that if m is a non-pseudo-atomic measure (in classic sense), defined on a σ-algebra $\mathcal{A}$ (of subsets of an abstract space T), and $A \in \mathcal{A}$ is an arbitrary set so that $m(A) > 0$, then for every element $b \in [0, m(A)]$, there exists a set $B \in \mathcal{A}$, so that $B \subseteq A$ and $m(B) = b$ (in other words, the set function m takes a continuum of values, and thus it does not omit any intermediate value).

V. A set function m is called *purely-atomic* if the space T can be represented as a finite or countable union of atoms of m.

Examples. (i) Let be the set $T = \{1, 2, ..., 9\}$. We define the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$ as follows: $\forall A \subseteq T, m(A) = \text{card}A$. Then $\forall i \in \{1, 2, ..., 9\}$, the singleton $\{i\}$ is an atom of m. Indeed, $\forall i \in \{1, 2, ..., 9\}$, $m(\{i\}) = 1 > 0$ and $\forall B \subseteq \{i\}$, we have either $B = \emptyset$, in which case $m(B) = 0$, or $B = \{i\}$ in which case $m(\{i\} \setminus B) = m(\emptyset) = 0$.

Consequently, in this case, any singleton is an atom.

(ii) Generally, *there is no relationship between the notion of an atom and that of a pseudo-atom*:

Let us consider an abstract set $T = \{t_1, t_2\}$ constituted of two distinct arbitrary elements and let also be the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$ defined for every $A \subset T$ by $m(A) = \begin{cases} 2, & \text{if } A = T \\ 1, & \text{if } A = \{t_1\} \\ 0, & \text{if } A = \{t_2\} \text{ or } A = \emptyset. \end{cases}$

Then T *is an atom and it is not a pseudo-atom for* m.

Indeed, $m(T) = 2 > 0$. Let be an arbitrary subset B of T.

If $B = \emptyset$, then $m(B) = 0$;

If $B = \{t_1\}$, then, by the definition, $m(T \setminus B) = m(\{t_2\}) = 0$;

If $B = \{t_2\}$, then, by the definition, $m(B) = 0$;

If $B = \{t_1, t_2\} (= T)$, then $m(T \setminus B) = m(\emptyset) = 0$.

Therefore, T is indeed an atom of m.

On the other hand, let us note that there exists the singleton $\{t_1\}$ for which $m(\{t_1\}) = 1 \neq 0$ and $m(\{t_1\}) = 1 \neq 2 = m(T)$. Consequently, T is not a pseudo-atom of m.

However, we note that, *if the set function m is null-addtive, then any atom of m is a pseudo-atom* (*).

Indeed, let us assume that $m : \mathcal{C} \to \mathbb{R}_+$ is a null-additive set function, and that the set $A \in \mathcal{C}$ is an atom of m. We shall prove that A is also a pseudo-atom of m.

Obviously, since A este atom, then $m(A) > 0$. Then, if we consider an arbitrary set $B \in \mathcal{C}$, with $B \subseteq A$, from the fact that A is an atom it follows that either $m(B) = 0$ or $m(A \setminus B) = 0$. In the latter case, because m is null-additive, it follows that $m(A) = m((A \setminus B) \cup B) = m(B)$. Consequently, A is a pseudo-atom of m.

Conversely, *if the set function* $m : \mathcal{C} \to \mathbb{R}_+$ is, moreover, finitely additive, then *any pseudo-atom* $A \in \mathcal{C}$ of m *is an atom*, too, and this immediately yields based on the equality $m(A) = m((A \setminus B) \cup B) = m(A \setminus B) + m(B) = m(B)$, which implies $m(A \setminus B) = 0$.

That is why, *in the framework of the classic measure theory* (a measure always possesses the null-additive property), *the notions of an atom and that of a pseudo-atom coincide.*

The converse of the above statement (*) does not generally hold since *there exist pseudo-atoms which are not atoms*, as the following example will show:

(ii) Let $T = \{t_1, t_2\}$ be an abstract set, containing two arbitrary elements, and let us consider the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$, defined for every set $A \subseteq T$, by $m(A) = \begin{cases} 1, if \ A \neq \emptyset \\ 0, \ if \ A = \emptyset. \end{cases}$

Then m is null-additive and $T = \{t_1, t_2\}$ is a *pseudo-atom of* m, *but it is not an atom of* m.

Let $A, B \subseteq T$ be so that $m(B) = 0$. By the definition of m we note that we must necessarily have $B = \emptyset$, whence $m(A \cup B) = m(A)$, and this proves that the set function m is null-additive.

We prove now that $T = \{t_1, t_2\}$ is a pseudo-atom of m. Indeed, we have $m(T) = 1 > 0$ and let $B \subseteq T$ an arbitrary subset.

If $B = \emptyset$, then $m(B) = 0$.

If $B \neq \emptyset$, then the set B either is a singleton, or is the set T, itself consisting of two elements. In both situations, one has $m(T) = 1 = m(B)$, which proves that $T = \{t_1, t_2\}$ is a pseudo-atom of m.

Let us prove now that $T = \{t_1, t_2\}$ is not an atom of m. Indeed, $m(T) = 1 > 0$ and there exists the singleton $\{t_1\}$ for which we have $m(\{t_1\}) = 1 \neq 0$ and $m(T \setminus \{t_1\}) = m(\{t_2\}) = 1 \neq 0$. Therefore, $T = \{t_1, t_2\}$ is not an atom of m.

(iii) The Dirac measure (or, the unit mass measure) (or, the δ-measure) δ_t concentrated in an arbitrary fixed point t of an abstract set T, is an example of a measure (in the classical sense) which is purely-atomic (Kadets, 2018).

The Dirac measure is defined as follows:

If $\mathcal{A}$ is a σ-algebra of subsets of T, then $\delta_t(A) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}, \forall A \in \mathcal{A}$.

Obviously, T is an atom of δ_t (because $\delta_t(T) = 1 > 0$ and $\forall A \in \mathcal{A}$, it holds either $\delta_t(A) = 0$ or $\delta_t(cA) = 0$, as $t \notin A$ or $t \in A$, that is, $t \notin cA$).

Let us recall now the following:

If $\mathcal{C}$ is a ring of subsets of an abstract space T and if $m : \mathcal{C} \to \mathbb{R}_+$ is a set function satisfying the condition $m(\emptyset) = 0$, two *sets* A_1, A_2 are said to be *equivalent* if $m(A_1 \triangle A_2) = 0$.

We note that if the set function m is additionally null-monotone and null-additive, then $m(A_1) = m(A_2)$ (which justifies the terminology, since the equivalence of the sets takes place in the sense of the "measurement" process).

Indeed, since $m(A_1 \triangle A_2) = m((A_1 \setminus A_2) \cup (A_2 \setminus A_1)) = 0$ and m is null-monotone, it follows that $m(A_1 \setminus A_2) = 0$ and $m(A_2 \setminus A_1) = 0$, whence, because m is null-additive and $m(A_1) = m((A_1 \setminus A_2) \cup (A_1 \cap A_2)) = m(A_1 \cap A_2)$, $m(A_2) = m((A_2 \setminus A_1) \cup (A_1 \cap A_2)) = m(A_1 \cap A_2)$, it follows that $m(A_1) = m(A_2)$.

We note that, *with respect to the Dirac measure* δ_t, *the atom* T *(the space itself, unreduced to a single point) is equivalent to the singleton* $\{t\}$, $t \in T$ (Kadets, 2018). Indeed, we have $m(T \triangle \{t\}) = 0$ (so, *with respect to the Dirac measure, the space "collapses" into a single point*).

We shall prove in the following that, *with respect to a monotone and null-additive set function, any set which is equivalent to an atom is itself an atom*:

Let us assume that the set A_1 is an atom and we prove that the set A_2, which is equivalent to the set A_1, possesses the same property. Indeed, according to the above statements, we have $m(A_2) = m(A_1) > 0$ and let $B \in \mathcal{C}$, $B \subseteq A$, be arbitrary.

If $m(B) = 0$, then the proof ends.

If $m(A_1 \setminus B) = 0$, then, since m is monotone and $m(A_1 \triangle A_2) = 0$, it follows that $m(A_2 \setminus A_1) = 0$.

On the other hand, again from the monotonicity of m we have $m(A_2 \setminus B) = m((A_2 \setminus A_1) \cup (A_1 \setminus B)) = m(A_1 \setminus B) = 0$, based also on the fact that m is null-additive and $m(A_2 \setminus A_1) = 0$. Consequently, $m(A_2 \setminus B) = 0$, and this finally proves that A_2 is an atom of m, too.

Let us also note that, with respect to a monotone and null-additive set function, *any set which is equivalent to a pseudo-atom is, itself, a pseudo-atom*:

Let us assume that the set A_1 is a pseudo-atom and we prove that the set A_2, which is equivalent to the set A_1, possesses the same property. Indeed, from the above statements, we have $m(A_2) = m(A_1) > 0$ and let $B \in \mathcal{C}$, $B \subseteq A$, be arbitrary.

If $m(B) = 0$, then the proof ends.

If $m(A_1) = m(B)$, then, since $m(A_2) = m(A_1) = m(B)$, it follows that A_2 is also a pseudo-atom of m.

5.5 Atoms and fractality

Next, we shall underline the fact that both the notion of atom and that of pseudo-atom (in the mathematical sense) possess a remarkable property, namely that of self-similarity (every part reflects the whole), a property which is a characteristic to fractals, both from a mathematical point of view and from the perspective of modern physics. This finding, among others, justifies the extension we illustrate in the last section, in which we address the necessity to introduce the notion of a fractal atom (Gavriluţ *et al.*, 2019).

The self-similarity property of the atoms (pseudo-atoms, respectively)

(i) If $m : \mathcal{C} \to \mathbb{R}_+$ is a null-monotone set function, with $m(\emptyset) = 0$, $A \in \mathcal{C}$ is an atom of m and $B \in \mathcal{C}$ is a subset of A having the property $m(B) > 0$, then B is also an atom of m and, moreover, $m(A \setminus B) = 0$ (which means that the "measure" of what remains when the set B is removed from the set A is null).

Indeed, one has $m(B) > 0$ and if we consider an arbitrary set $C \in \mathcal{C}$, with $C \subseteq B$, then, since $B \subseteq A$, it follows that $C \subseteq A$.

If $m(C) = 0$, the proof ends.

Let us assume now that $m(C) \neq 0$. Because $A \in \mathcal{C}$ is an atom of m, it follows that $m(A \setminus C) = 0$.

Since $B \setminus C \subseteq A \setminus C$ and m is null-monotone it gets that $m\,(B \setminus C) = 0$ and, therefore, B is an atom of m.

Moreover, since $A \in \mathcal{C}$ is an atom of m and $B \in \mathcal{C}$ is a subset satisfying the property $m(B) > 0$, then we must necessarily have $m(A \setminus B) = 0$.

(ii) If $A \in \mathcal{C}$ is a pseudo-atom of m and the set $B \in \mathcal{C}$ satisfies $B \subseteq A$ and $m(B) > 0$, then B is also a pseudo-atom of m and, moreover, $m(B) = m(A)$ (which means that the sets A are B are "identical" with respect to the "measure" m).

Indeed, we have $m(B) > 0$ and, if we consider an arbitrary set $C \in \mathcal{C}$, with $C \subseteq B$, then, since $B \subseteq A$, it follows that $C \subseteq A$.

If $m(C) = 0$, the proof ends.

Let us assume now that $m(C) \neq 0$. Since $A \in \mathcal{C}$ is a pseudo-atom of m, it follows that $m(A) = m(C)$.

On the other hand, since $A \in \mathcal{C}$ is a pseudo-atom of m, the set $B \in \mathcal{C}$ satisfies $B \subseteq A$ and $m(B) > 0$, then $m(B) = m(A)$.

In consequence, $m\,(B) = m(C)$, and this finally proves that B is also a pseudo-atom of m.

Let us make, at the end of this section, the following observation:

Assuming that a set function $m : \mathcal{C} \to \mathbb{R}_+$ is monotone, null-additive and regular (meaning that, roughly speaking, we can, through it, approximate sets about which we have little information, with sets about which we have more information), one can prove that for each atom A of m (if it exists), there exists a unique element $a \in A$ so that $m(A) = m(\{a\})$ (Pap, 1995) (this means that the *"measure" of the atom is equal to the measure of each "point" it contains, and this reflects the holographic perspective, according to which the information is concentrated in a single point.*

Indeed, any part of a fractal reflects the whole, as a pattern:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

(William Blake, *Auguries of Innocence*)

5.6 Minimal atoms

We shall now introduce a very special category of atoms, which we show to reflect the property of indivisibility (non-decomposability).

Let $\mathcal{C}$ be an arbitrary ring of subsets of an abstract space T and let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function so that $m(\emptyset) = 0$.

A set $A \in \mathcal{C}$ is called a *minimal atom* of m if $m(A) > 0$ and for every subset $B \in \mathcal{C}$ ($B \subseteq A$) it holds either $m(B) = 0$, or $B = A$ (Ouyang *et al.*, 2015).

In other words, a minimal atom is a special set, of stricly positive "measure", so that any of its subsets has either zero "measure", or identifies with the set itself. Thus, a minimal atom has the property that any of its subsets has either zero "measure" (that is, it is negligible during the "measurement" process), or identifies with the initial set (without the need of a "measurement" process).

Let us note that the terminology is justified. Indeed, if $A \in \mathcal{C}$ is a minimal atom of m, then for m there cannot exist other minimal atom $A_1 \in \mathcal{C}$, which is different from A and satisfies $A_1 \subset A$.

Indeed, if we assume, on the contrary, that there exists another minimal atom $A_1 \in \mathcal{C}$ which is different from A and satisfies $A_1 \subset A$, then, since A_1 is a minimal atom, we get that $m(A_1) > 0$. Because $A_1 \subsetneq A$, then $A_1 = A$, and this is false due to the assumption we made.

Example. Let $T = \{a, b, c, d\}$ be an abstract set, constituted of four distinct elements and let also be the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$, defined for every $A \subseteq T$ by $m(A) = \begin{cases} 5, & \text{if } A = T \\ 2, & \text{if } A \neq T \\ 0, & \text{if } A = \emptyset. \end{cases}$

We note that any singleton (i.e., a set containing only one element) is a minimal atom of m. Indeed, the "measure" m of any singleton is, according to the definition, 2, so it is strictly positive and any subset is either void and hence has zero measure, or is the set itself.

We note that, in general, *any minimal atom is, particularly, an atom and also a pseudo-atom.*

Indeed, if $A \in \mathcal{C}$ is a minimal atom of m, then $m(A) > 0$ and for any of its subset $B \in \mathcal{C}$ $(B \subseteq A)$ it holds either $m(B) = 0$, or $B = A$. The latter posibility yields $m(A \setminus B) = 0$ and $m(B) = m(A)$, so A is both an atom and a pseudo-atom of m.

The following examples highlight the fact that there exists generally no relationship between the notions of atom/pseudo-atom and that of minimal atom:

Examples. (i) Let $T = \{a, b\}$ be an abstract set constituted of two distinct elements and let also be the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$ defined as follows:

$$\forall A \subseteq T, m(A) = \begin{cases} 1, & \text{if } A = \{a\} \text{ or } A = T \\ 0, & \text{otherwise.} \end{cases}$$

Then T *is an atom of* m:

Obviously, $m(T) = 1 > 0$. Let $B \subseteq T$ be an arbitrary set.

If $B = \emptyset$, then $m(B) = 0$.

If $B = \{a\}$, then $m(T \setminus B) = m(\{b\}) = 0$.

If $B = \{b\}$, then $m(B) = 0$.

If $B = T = \{a, b\}$, then $m(T \setminus B) = m(\emptyset) = 0$.

But T *is not a minimal atom of* m:

Obviously, one has $m(T) = 1 > 0$ and let $B \subseteq T$ be an arbitrary set. We observe that there exists the set $B = \{a\} \neq T$ for which $m(B) = 1 \neq 0$.

We also note that *the set* $\{a\}$ *is an atom* (we have $m(\{a\}) = 1 > 0$ and any subset $B \subseteq \{a\}$ either is void, so $m(B) = 0$, or is the set $\{a\}$ itself, so $m(\{a\} \setminus \{a\}) = 0$). The *set* $\{a\}$ *is also a minimal atom* of m since $m(\{a\}) = 1 > 0$ and any subset $B \subseteq \{a\}$ either is void, so $m(B) = 0$, or is $\{a\}$ itself.

(ii) Let $T = \{a, b, c, d\}$ be an abstract set constituted of four distinct elements and let also be the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$, defined as follows: $\forall A \subseteq T$,

$$m(A) = \begin{cases} 5, & \text{if } A = T \\ 3, & \text{if } A = \{a, b, c\} \text{ or } A = \{a, b, d\} \text{ or } A = \{a, c, d\} \\ 2, & \text{if } A = \{a, b\} \text{ or } A = \{a, c\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{a, b\}$ and $\{a, c\}$ *are minimal atoms of* m. We shall prove the statement, for instance, for the $\{a, b\}$:

Indeed, we have $m(\{a, b\}) = 2 > 0$ and let B be an arbitrary subset.

If $B = \{a, b\}$, the statement is verified.

If $B = \{a\}$ or $B = \{b\}$, then, according to the definition, we have $m(\{a\}) = m(\{b\}) = 0$, so the statement is again verified.

If $B = \emptyset$, then $m(B) = 0$.

In the following, let us note that if $m : \mathcal{C} \to \mathbb{R}_+$ is a null-null-additive set function and $A, B \in \mathcal{C}$ are two different minimal atoms of m, then they must be necessarily disjoint, that is, $AnB = \emptyset$.

Indeed, let us assume that, on the contrary, $A \cap B \neq \emptyset$. Since $A, B \in \mathcal{C}$ are two minimal atoms of m, $A \setminus (A \cap B) = A \setminus B \subseteq A$ and $A \cap B \subseteq B$, it follows that $[m(A \setminus B) = 0$ or $A \setminus B = A]$ and $[m(A \cap B) = 0$ or $A \cap B = B]$.

(i) If $A \setminus B = A$, then $A \cap B = \emptyset$, which is false since, according to our assumption, we have $A \cap B \neq \emptyset$.

(ii) If $m(A \setminus B) = 0$ and $m(A \cap B) = 0$, then, since m is null-null-additive, one gets that $m(A) = m((A \setminus B) \cup (A \cap B)) = 0$, which is false, since $m(A) > 0$, the set A being a minimal atom of m.

(iii) If $m(A \setminus B) = 0$ and $A \cap B = B$, then $B \subseteq A$, so, since A is a minimal atom of m, one gets from the above observation that $B = A$, which is false.

Consequently, $A \cap B = \emptyset$.

The property we shall demonstrate next reflects *the non-decomposability (non-partitionability) of the minimal atoms:*

A minimal atom $A \in \mathcal{C}$ of a null-null-additive set function m cannot be partitioned in sets that are elements of $\mathcal{C}$. Indeed, if we suppose, on the contrary, that there exists a partition of a lui A, this means that there exists a family $\{A_i\}_{i \in \{1,2,\ldots,p\}}$ of nonvoid sets of $\mathcal{C}$ so that $\cup_{i=1}^{p} A_i = A$ and the sets A_i are pairwise disjoint.

Referring to the first set A_1, since $A \in \mathcal{C}$ is a minimal atom, it follows that we cannot have the situation $A_1 = A$. Therefore, $m(A_1) = 0$. Analogously, for the second set, A_2, we get that $m(A_2) = 0$. Recurrently, it gets that $m(A_3) = ... = m(A_p) = 0$. Since m is null-null-additive, it follows that $m(A) = m(\cup_{i=1}^{p} A_i) = 0$, which is obviously false.

Consequently, any minimal atom is non-decomposable.

In the following, we shall prove that the converse of this statement also holds, namely, we shall demonstrate that *any non-decomposable atom $A \in \mathcal{C}$ is necessarily a minimal atom.* Indeed, since the set A is an atom, then $m(A) > 0$.

Since the set A is not partitionable, there cannot exist two nonvoid disjoint subsets $A_1, A_2 \in \mathcal{C}$ of A so that $A = A_1 \cup A_2$.

Let be an arbitrary set $B \in \mathcal{C}$, with $B \subseteq A$.

If $m(B) = 0$, then the proof ends.

If $m(B) > 0$, since $B \subseteq A$, one gets that $B = A$ (otherwise, the family $\{A \setminus B, B\}$ is a partition of A: $A \setminus B, B \in \mathcal{C}$, $(A \setminus B) \cap B = \emptyset$, $(A \setminus B) \cup B = A$, which is false).

Consequently, A is a minimal atom.

From the two statements above, one arrives at the following conclusion: *an atom is minimal if and only if it is not partitionable (it is non-decomposable).*

In the following, we shall highlight the fact that, *in the case when the abstract set T is finite*, then *any set $A \in \mathcal{C}$, satisfying the condition $m(A) > 0$ possesses at least one set $B \in \mathcal{C}$, $B \subseteq A$, which is a minimal atom minimal of m.*

Moreover, in the particular case when A is an atom of m and the set function m is null-additive, one gets that $m(A) = m(B)$ and the set B is unique.

Indeed, let us consider the family of sets $\mathcal{M} = \{M \in \mathcal{C},\ M \subseteq A, m(M) > 0\}$. Obviously, since $A \in \mathcal{C}$, then $\mathcal{M} \neq \emptyset$.

We note that any minimal element $M \in \mathcal{M}$ of $\mathcal{M}$ is a minimal atom of m. Indeed, since M is a minimal element, there cannot exist another set $D \in \mathcal{M}$ so that $D \subsetneq M$ (**).

Since $M \in \mathcal{M}$, this means that $M \in \mathcal{C}$, $M \subseteq A$ and $m(M) > 0$.

We shall prove that M is a minimal atom of m. Indeed, for any set $S \subseteq M, S \in \mathcal{C}$, we have either $m(S) = 0$ or $m(S) > 0$. In the latter case, we have either $S = M$ (which is suitable) or $S \neq M$, which contradicts the statement $(**)$.

Let us assume, moreover, that the set A is an atom of m and m is null-additive. According to the considerations proved above, there exists at least one set $B \in \mathcal{C}$, $B \subseteq A$, which is a minimal atom of m. This means that $m(B) > 0$ and, because A is an atom, we must necessarily have $m(A \setminus B) = 0$. Since m is null-additive, this yields $m(A) = m((A \setminus B) \cup B) = m(B)$.

It only remains to prove that the set B is unique. Indeed, if we suppose, on the contrary, that there exist two different minimal atoms B_1 and B_2 of m, this would imply, as before, that $m(A \setminus B_1) = m(A \setminus B_2) = 0$.

If $m(B_1 \cap B_2) = 0$, then $m(A) = m(A \setminus (B_1 \cap B_2) \cup (B_1 \cap B_2)) = m(A \setminus (B_1 \cap B_2)) = m((A \setminus B_1) \cup (A \setminus B_2))$, which is false.

If $m(B_1 \cap B_2) > 0$, since B_1 and B_2 are minimal atoms of m, it results that $B_1 = B_1 \cap B_2 = B_2$, which is again false.

Finally, we shall prove that, *if the set T is finite*, the set function m is null-additive, and $\{A_i\}_{i \in \{1,2,\ldots,p\}}$ is the family of all different minimal atoms which are contained in a set $A \in \mathcal{C}$, satisfying $m(A) > 0$ (we proved in the above considerations that such atoms exist), then $m(A) = m\left(\cup_{i=1}^{p} A_i\right)$.

(This means that *the set A identifies itself, from the "measure" m viewpoint, with the union of all different minimal atoms which it contains*, therefore the *minimal atoms are the only ones that matter from the "measurement" point of view)*.

Let us note that $m(A \setminus \cup_{i=1}^{p} A_i) = 0$ (if, on the contrary, one has $m(A \setminus \cup_{i=1}^{p} A_i) > 0$, from the statement proved above it would follow that there exists at least one set $B \in \mathcal{C}$, $B \subseteq A \setminus \cup_{i=1}^{p} A_i \subseteq A$, which is a minimal atom of m, and this is false since $A_1, ..., A_p$ are the only different minimal atoms contained in A).

Since $m(A \setminus \cup_{i=1}^{p} A_i) = 0$ and m is null-additive, it follows that $m(A) = m((A \setminus \cup_{i=1}^{p} A_i) \cup (\cup_{i=1}^{p} A_i)) = m(\cup_{i=1}^{p} A_i)$.

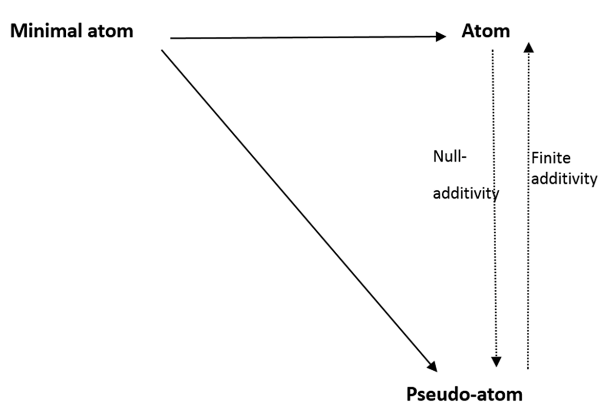
Let us finally note the following:

1. Any minimal atom is also an atom and a pseudo-atom (which justifies the terminology);

2. If the set function is null-additive, then any of its atoms is a pseudo-atom, too;

3. If, moreover, the set function is finitely additive, then the converse of the above statement is also valid, therefore any pseudo-atom is particularly an atom.

Consequently, for a finitely additive set function (which automatically possesses the null-additivity property), the notion of atom and that of pseudo-atom coincide. We summarize all these observations in the following schematic:



6 Extensions of the notions of atom

Generalizations of the mathematical notion of an atom have been made, so far, in two major directions. A first direction is given by the fact that, instead of set functions, which are indispensable to the process of the so-called "measurement", one could generally operate with set multifunctions (that is, functions that associate a set to another set). Thus, results with a higher degree of generalization and abstraction can be obtained. The second direction is given by the correlation that can be made by placing the notion of (minimal) atom within the fractal sets theory, thus resulting in the notion of *fractal (minimal) atom* (Gavrilut *et al.*, 2019; Gavrilu̧t and Agop, 2016).

It is no coincidence that in literature, Eminescu's manuscript notebooks contain observations on the holographic principle, which constitutes the essence of interrelation, of interconnection: "In the ether, each atom is an individual which is connected with everything".

In his *Memoirs*, Werner Heisenberg recounts that in the summer of 1920, Niels Bohr, one of the fathers of quantum theory, told him that: "When we talk about atoms, language can be used just as in poetry. The poet is not so much concerned with the description of facts, as with the creation of images and the establishment of mental connections". A language problem that Bruce Rosenblum and Fred Kuttner reprised under another guise in their book, *The quantum enigma*: "In quantum theory there is no atom in addition to the wave function of the atom. This is so crucial that we say it again in other words. The atom's wave-functions and the atom are the same thing; *the wave function of the atom* is a synonym for *the atom.*" (2011).

6.1 Set multifunctions

A triplet $(X, +, \cdot)$ constituted of an abstract, nonvoid set X, an operation "+" of addition on X and an operation "$\cdot$" of multiplication with real scalars is called a *real linear space on* $\mathbb{R}$ if the following axioms are fulfilled:

1) associativity: $x + (y + z) = (x + y) + z$, $\forall x, y, z \in X$;

2) the existence of the identity element of addition θ (called the *origin of the space* X): $\exists \theta \in X$ so that $\forall x \in X$, $x + \theta = \theta + x = x$;

3) $\forall x \in X$, $\exists - x \in X$ (called the additive inverse of x) so that $x + (-x) = (-x) + x = \theta$;

4) commutativity: $x + y = y + x$, $\forall x, y \in X$;

5) $\lambda(x + y) = \lambda x + \lambda y$, $\forall x, y \in X$, $\forall \lambda \in \mathbb{R}$;

6) $(\lambda + \mu)x = \lambda x + \mu x$, $\forall x \in X$, $\forall \lambda, \mu \in \mathbb{R}$;

7) $\lambda(\mu x) = (\lambda\mu)x$, $\forall x \in X$, $\forall \lambda, \mu \in \mathbb{R}$;

8) $1 \cdot x = x$, $\forall x \in X$.

For instance, the set $\mathbb{R}$ of real numbers, endowed with the operation "+" of addition of two real numbers and also with the operation "$\cdot$" of multiplication of a real number with a real scalar, that is, the triplet $(\mathbb{R}, +, \cdot)$, constitutes a real linear space.

If $(X, +, \cdot)$ is a real linear space, then a function $\|\cdot\| : X \to \mathbb{R}_+$ is called a *norm* on the space X if the following axioms are fulfilled:

$$N_1 \, \|x\| \geq 0, \ \forall x \in X; \|x\| = 0 \Leftrightarrow x = \theta \text{ (the positivity of the norm);}$$

$$N_2 \, \|\lambda x\| = |\lambda| \cdot \|x\| \,, \ \forall x \in X, \forall \lambda \in \mathbb{R} \text{ (the homogeneity of the norm);}$$

$$N_3 \|x{+}y\| = \|x\|{+}\|y\|, \ \forall x, y \in X \text{ (the triangular inequality).}$$

The pair $(X, \|\cdot\|)$ is called a *normed space*.

For instance, $(\mathbb{R}, +, \cdot)$ endowed with the norm defined by $\|x\| = |x|$, $\forall x \in \mathbb{R}$, is a real normed space.

In what follows, let be an abstract nonvoid set T, $\mathcal{C}$ a ring of subsets of T, X a real linear normed space with the origin θ and $\mathcal{P}_0(X)$, the family of all nonvoid subsets of X.

By a *set multifunction* we mean a function (or, application) which associates a set to another set, in contrast with the notion of a function, which associates a point to another point.

So, in what follows, let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be an arbitrary set multifunction satisfying the condition $\mu(\emptyset) = \theta$.

The notions of atom, pseudo-atom, minimal atom introduced with respect to a set function m can be generalized in this context, with respect to the set multifunction μ, as follows:

We say that a set $A \in \mathcal{C}$ is:

(i) an *atom* of μ if $\mu(A) \supsetneq \{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A$, we have either $\mu(B) = \{\theta\}$ or $\mu(A \setminus B) = \{\theta\}$;

(ii) a *pseudo-atom* of μ if $(A) \supsetneq \{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A$, it holds either $\mu(B) = \{\theta\}$, or $\mu(A) = \mu(B)$;

(iii) a *minimal atom of* μ if $\mu(A) \supsetneq \{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A$, one has either $\mu(B) = \{\theta\}$ or $A = B$.

Detailed considerations on the problem of atomicity with respect to set multifunctions can be found, for instance, in Gavriluţ and Agop, 2016, and also in Gavriluţ *et al.* 2019.

6.2 Towards a fractal theory of atomicity

The main idea in the quantum theory of measure and in generalized quantum mechanics is to provide a description of the world in terms of histories. A history is a classical description of the system considered for a certain period of time, which may be finite or infinite. If one tries to describe a particle system, then a history will be given by classical trajectories. If one deals with a field theory, then a history corresponds to the spatial configuration of the field as a function of time.

In both cases, the quantum theory of measure tries to provide a way to describe the world through classical histories, extending the notion of probability theory, which is obviously not enough to shape our universe.

On the other hand, ordinary structures, self-similar structures etc. of nature can be assimilated to complex systems, if one considers both their structure and functionality (Gavriluţ and Agop, 2013).

The models used to study the complex systems dynamics are built on the assumption that the physical quantities that describe it (such as density, momentum, and energy) are differentiable. Unfortunately, differentiable methods fail when reporting to physical reality, due to instabilities in the case of complex systems dynamics, instabilities that can generate both chaos and patterns.

In order to describe such dynamics of the complex systems, one should introduce the scale resolution in the expressions of the physical variables describing such dynamics, as well as in the fundamental equations of the evolution (density, kinetic moment and equations of the energy). This way, any dynamic variable which is dependent, in a classical sense, both on the space and time coordinates, becomes, in this new context, dependent on scale resolution as well.

Therefore, instead of working with a dynamic variable, we can deal with different approximations of a mathematical function that is strictly non-differentiable. Consequently, any dynamic variable acts as the limit of a family of functions. Any function is non-differentiable at a zero resolution scale and it is differentiable at a non-zero resolution scale.

This approach, well adapted for applications in the field of complex systems dynamics, in which any real determination is made at a finite resolution scale, clearly involves the development of both a new geometric structure and a physical theory (applied to the complex systems dynamics) for which the motion laws, that are invariant to the transformations of spatial and temporal coordinates, are integrated with scale laws, which are invariant to transformations of scale.

Such a theory that includes the geometric structure based on the assumptions presented above was developed in the scale relativity theory and, more recently, in the scale relativity theory with constant arbitrary fractal dimension. Both theories define the class of fractal physics models.

In this model, it is assumed that, in the complex systems dynamics, the complexity of interactions is replaced by non-differentiability. Also, the motions forced to take place on continuous, differentiable curves in a Euclidean space are replaced by free motions, without constraints, that take place on continuous, non-differentiable curves (called fractal curves) in a fractal space (Agop *et al.*, 2019).

In other words, for a time resolution scale that seems large when compared to the inverse of the largest Lyapunov exponent, deterministic trajectories can be replaced by a set of potential trajectories, so that the notion of "defined positions" is replaced by the concept of a set of positions that have a definite probability density.

In such a conjecture, quantum mechanics becomes a particular case of fractal mechanics (for the structural units motions of a complex system on Peano curves at Compton scale resolution).

Therefore, the quantum theory of the measure could become a particular case of a fractal measure theory.

One of the concepts that needs to be defined is that of a fractal minimal atom, as a generalization of the concept of a minimal atom (for details refer to Gavriluţ *et al.*, 2019).

Current research in mathematics and physics brings us closer to the ultimate structure of matter and the understanding that, on the one hand, the notion of identifiable particle is becoming increasingly irrelevant, this kind of divisions and sequencing being in fact operative notions required for experiments and gradual understanding, rather than natural elements; on the other hand, that this approach to the core, to the essence, reveals laws of overwhelming simplicity, and that mathematics, also scattered, in the last hundred years, in structures and increasingly luxuriant and complex formulas, finds thus its role as an agile, beautiful, universal language through which nature expresses itself, as Galileo used to say.

Not only nature expresses itself thus, art does, too! Here is just one example of the decomposition / recomposition of a classic masterpiece, in the vision of Salvador Dali (see Figure 1 below):

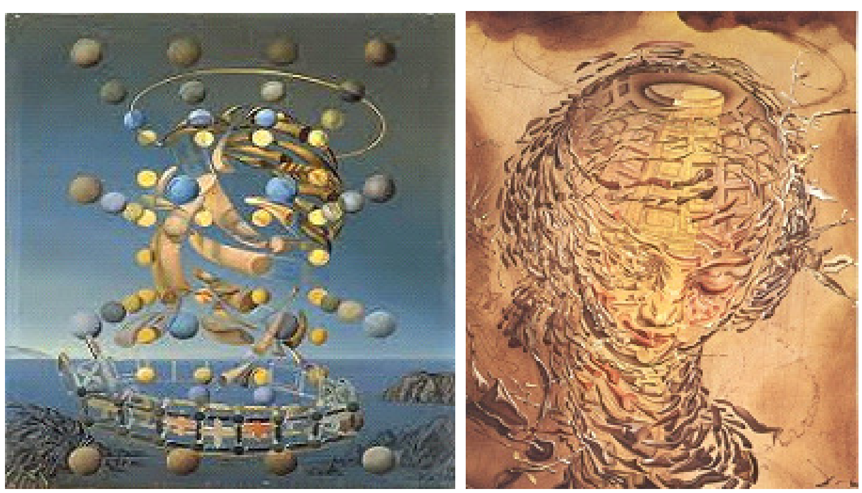


Figure 1: *Dali's Maximum Speed of Raphael's Madonna and Raphaelesque Head Exploding stage objects or suspended fragments of objects reminiscent of electrons around an atom, about to aggregate into an object, character, figure or, conversely, to decompose into an image constructed according to the rigors of the golden ratio.*

Let us leave Steven Weinberg the - provisionally - final word of this mathematical overview on the atom and its integrative relationship with the other fields of thought and expression:

> There is one clue in today's elementary particle physics that we are not only at the deepest level we can go to right now, but that we are at the level which is in fact in absolute terms quite deep, perhaps close to the final source.

> There is reason to believe that in elementary particle physics we are learning something about the logical structure of the universe at a very, very deep level. The reason I say this is that as we have been going to higher and higher energies and as we have been studying structures that are smaller and smaller we have found that the laws, the physical principles, that describe what we learn become simpler and simpler.
> I'm not saying that the mathematics gets easier, Lord knows it doesn't. I'm not saying we always find fewer particles in our list of elementary particles. What I am saying is that the rules we have discovered are becoming increasingly coherent and universal. We are beginning to suspect that this isn't just an accident of the particular problems we have chosen to study at this moment in the history of physics, but that there is simplicity, a beauty, that we are finding in the rules that govern matter that mirrors something that is built into the logical structure of the universe at a very deep level.
>
> (Weinberg, Facing Up, 210)

7 Conclusions

After having presented the notion of the atom from a multidisciplinary and even transdisciplinary perspective, we focused on detailing the problem of atomicity from a mathematical perspective, emphasising the fact that an atom is a mathematical object which, in essence, has no other subobjects than the object itself or the null subobject. We have also tried to demonstrate that, in its various forms and mathematical meanings of being, an atom is, essentially, indestructible, indivisible, irreducible and self-similar.

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References

[1] M. Agop, A. Gavriluţ, L. Eva, G. Crumpei, *Towards the Multifractal Brain by means of the Informational Paradigm. Fundaments and Applications*, ArsLonga Publishing House, 2019, ISBN: 978-973-148-328-3.

[2] D. Bohm, *Wholeness and the Implicate Order*, Routledge, 1980.

[3] B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 2002.

[4] M.A. Fike, *The One Mind. C.G. Jung and the Future of Literary Criticism*, Routledge, Francis and Taylor Group, London and New York, 2014.

[5] D.H. Fremlin, *Measure Theory*, Vol. 5, 2000.

[6] A. Gavriluţ, M. Agop, *A Mathematical Approach in the Study of the Dynamics of Complex Systems* (in Romanian), ArsLonga Publishing House, Iaşi, 2013.

[7] A. Gavriluţ, M. Agop, *An Introduction to the Mathematical World of Atomicity through a Physical Approach*, ArsLonga Publishing House, Iaşi, 2016.

[8] A. Gavriluţ, I. Mercheş, M. Agop, *Atomicity through Fractal Measure Theory: Mathematical and Physical Fundamentals with Applications*, Springer, 2019, DOI: 10.1007/978-3-030-29593-6, ISBN: 978-3-030-29592-9.

[9] G. Gavriluţ, A. Gavriluţ, M. Agop, *Extended Minimal Atomicity through Non-differentiability: A Mathematical-Physical Approach*, Advances in Mathematical Physics 2019 (160), 1-16, DOI: 10.1155/2019/8298691.

[10] T. Gyatso (the XIVth Dalai Lama), *The Universe in a Single Atom: The Convergence of Science and Spirituality*, Humanitas Publishing House, Bucureşti, 2011.

[11] Hudson H. (ed.), 2004, *Simples*, special issue of The Monist 87: 303451.

[12] C.G. Jung, *The Pauli/Jung Letters*, 1932-1958, Edited by C.A. Meier Princeton U. Press, Princeton, N.J., 2001.

[13] V. Kadets, *A Course in Functional Analysis and Measure Theory*, Springer, 2018.

[14] K. Koffka. *Principles of Gestalt Psychology*, New York: Harcourt-Brace, 1935.

[15] I. Marshall, D. Zohar, *Who's Afraid of Schrödinger's Cat? The New Science Revealed, Quantum Theory, Relativity, Chaos and the New Cosmology*, Bloomsbury Publishing, 1997.

[16] Y. Ouyang, J. Li, R. Mesiar, *Relationship between the concave integrals and the pan-integrals on finite spaces*, Journal of Mathematical Analysis and Applications, Vol. 424, No. 2, 975-987, 2015.

[17] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, 1995.

[18] E. Pap, *Regular null-additive monotone set functions*, Univ. u Novom Sadu, Zb. Rad. Prirod. - Mat. Fak., Ser. Mat. 25, 2 (1995), 93-101.

[19] A.M. Precupanu, *The Basis of the Mathematical Analysis* (in Romanian), Polirom Publishing House, Iaşi, 1998.

[20] A.M. Precupanu, *Mathematical Analysis. Measure and Integral* (in Romanian), Al.I. Cuza University Publishing House, 2006.

[21] B. Rosenblum, F. Kuttner, *Quantum Enigma: Physics Encounters Consciousness*, Oxford University Press, 2006.

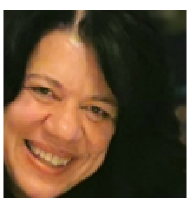
[22] H.L. Royden, *Real Analysis* (3rd ed.), New York: Macmillan, 1988.

[23] J.C. Smuts, *Holism and Evolution*, The MacMillan Company, 1926.

[24] H.-C.Tsai, *Infinite "Atomic" Mereological Structures*, Philosophical Logic: Current Trends in Asia, 2017, 287-296.

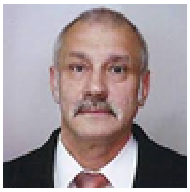
[25] A.C. Varzi, *Spatial Reasoning and Ontology: Parts, Wholes and Locations*, chapter published in M. Aiello, I. Pratt-Hartmann, J. van Benthem (eds.), Handbook of Spatial Logics, Berlin: Springer-Verlag, 2007, 945-1038.

About the Authors



Simona MODREANU, university professor, director of the Department of French language and literature at the Alexandru Ioan Cuza University in Iasi (Romania), translator. PhD from the University of Paris VII. Director of the Romanian Cultural Center in Paris (1999-2001). Author of the books: Cioran or the forbidden prayer (Iasi, 2002); Eugène Ionesco or the agony of significance (Iasi, 2002); The Paradoxical God of Cioran (Paris, 2003); Cioran (Paris, OXUS, 2003); Nomadic Readings (Iaşi, 2006); Sedentary Readings (Iasi, 2010), Unfaithful Readings (Iasi, 2014), Fluid Readings (Iasi, 2018), The Atom, an endless story (Iasi, 2020); Cioran or the chance of failure (Paris, 2021).

Alina GAVRILUȚ is a PhD lecturer in the Department of Mathematics at Al.I. Cuza University of Iasi. She is editor and member of the Editorial Board of several international mathematics journals. Her areas of interest are theoretical and applied mathematical analysis, focusing on the theory of classical and non-additive (set) measurements, general topology and hypertopologies, nonlinear dynamics. She has published over 50 articles in ISI journals, 11 books, and has given numerous papers at national and international scientific events.



Maricel AGOP is a professor in the Department of Physics at Gh. Asachi Technical University of Iași. His main area of interest is the use of nonlinear dynamics in the study of different physical systems (plasma, nanostructures, composite). In recent years, he has turned his attention to biological systems, especially those related to medicine (solid tumors, cancer cells) and the pharmaceutical field (nano-structured drug delivery systems). He has published over 260 articles in ISI journals and 90 books in Romania and abroad. Since 2018, he is a corresponding member of the Romanian Academy of Scientists.

Gabriel CRUMPEI graduated from the Faculty of General Medicine of the University Gr.T. Popa and the Faculty of Theology of Alexandru Ioan Cuza University. He specialized in psychiatry and received his doctorate in psychology in 2003. He is the director of a Center for Psychiatry, Psychotherapy and Counseling. He is also an associate professor at the Faculty of Psychology of Alexandru Ioan Cuza University in Iasi. His main interests are psychology, psychiatry, theology, physics and neurosciences. He is the author and co-author of several ISI papers, books and book chapters and has given numerous papers at national and international scientific events.